Problem Session 9 03/27/2019

(1) Problem 7.8, Jackson.

(2) Problem 7.9, Jackson.

(1) (a) The layer matrix is:

$$\mathbb{P}^{(j)} = \begin{bmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{bmatrix} = \cos(k_j t_j) \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbb{1}} + i\sin(k_j t_j) \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\sigma_3}$$

$$= \boxed{\mathbb{1}\cos(k_j t_j) + i\sigma_3 \sin(k_j t_j)}$$

(b) The interface matrix $\mathbb{I}_{1\to 2}$ is:

$$\mathbb{I}_{1\to 2} = \begin{bmatrix} \frac{1}{t_{21}} & \frac{r_{21}}{t_{21}} \\ \frac{r_{21}}{t_{21}} & \frac{1}{t_{21}} \end{bmatrix}$$

Where:

$$r_{21} = \frac{n_2 - n_1}{n_2 + n_1} = \frac{1 - \frac{n_1}{n_2}}{1 + \frac{n_1}{n_2}}, \quad t_{21} = \frac{2n_2}{n_2 + n_1} = \frac{2}{1 + \frac{n_1}{n_2}}$$

Thus, using $n \equiv \frac{n_1}{n_2}$, we have:

$$\Pi_{1\to 2} = \frac{1}{2}\begin{bmatrix} 1+n & 1-n \\ 1-n & 1+n \end{bmatrix} = \frac{1}{2}(n+1)\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbb{1}} - \frac{1}{2}(n-1)\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\sigma_1}$$

$$\boxed{= \mathbb{1}\left(\frac{n+1}{2}\right) - \sigma_1\left(\frac{n-1}{2}\right)}$$

(c) We have:

$$\begin{bmatrix} E_{trans} \\ 0 \end{bmatrix} = \Pi \begin{bmatrix} E_{inc} \\ E_{refl} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}\begin{bmatrix} E_{inc} \\ E_{refl} \end{bmatrix} \Rightarrow$$

$$\begin{cases} t_{11} E_{inc} + t_{12} E_{refl} = E_{trans} \\ t_{21} E_{inc} + t_{22} E_{refl} = 0 \end{cases}$$

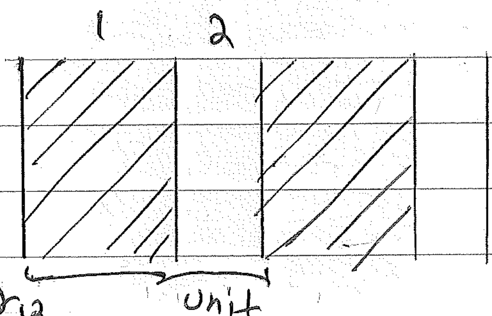
The second equation gives:

$$\boxed{E_{refl} = -\frac{t_{21}}{t_{22}} E_{inc}}$$

Substituting this in the first equation yields:

$$t_{11} E_{inc} - \frac{t_{12} t_{21}}{t_{22}} E_{inc} = E_{trans} \Rightarrow \boxed{E_{trans} = \frac{\det(\Pi)}{t_{22}} E_{inc}}$$

(2) For each unit, the transfer matrix is given by:



$$\mathbb{T}_{unit} = \begin{bmatrix} e^{i\alpha_2} & 0 \\ 0 & e^{-i\alpha_2} \end{bmatrix} \begin{bmatrix} \frac{1}{t_{21}} & \frac{r_{21}}{t_{21}} \\ \frac{r_{21}}{t_{21}} & \frac{1}{t_{21}} \end{bmatrix} \begin{bmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{-i\alpha_1} \end{bmatrix} \begin{bmatrix} \frac{1}{t_{12}} & \frac{r_{12}}{t_{12}} \\ \frac{r_{12}}{t_{12}} & \frac{1}{t_{12}} \end{bmatrix}$$

Where:

$$r_{21} = \frac{1-n}{1+n}, \quad t_{21} = \frac{2}{1+n}, \quad t_{12} = \frac{2n}{1+n}, \quad r_{12} = \frac{n-1}{n+1} = -r_{21}$$

This results in:

$$\mathbb{T}_{unit} = \frac{(1+n)^2}{4n} \left(\cos\alpha_2 \, \mathbb{1} + i\sin\alpha_2 \, \sigma_3\right)\left(1 - r\sigma_1\right)\left(\cos\alpha_1 \, \mathbb{1} + i\sin\alpha_1 \, \sigma_3\right)\left(1 + r\sigma_1\right)$$

$$r \equiv \frac{n-1}{n+1}$$

After introducing the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$, and using $\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k$, we find:

$$\mathbb{T}_{unit} = \frac{(1+n)^2}{4n} \left[\cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2 - r^2 \cos\alpha_1 \cos\alpha_2 - r^2 \sin\alpha_1 \sin\alpha_2\right]\mathbb{1}$$

$$- r\left(\sin\alpha_1 \sin\alpha_2 + \cos\alpha_1 \cos\alpha_2\right)\sigma_1 - 2r\sin\alpha_1 \cos\alpha_2 \, \sigma_2 + \left[i\sin\alpha_1 \cos\alpha_2\right.$$

$$+ i\sin\alpha_2\cos\alpha_1 + i r^2 \sin\alpha_1\cos\alpha_2 - i r^2 \sin\alpha_2 \cos\alpha_1 \big]\, \sigma_3$$

Note that $\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2 = \cos(\alpha_1+\alpha_2)$ and $\sin\alpha_1\cos\alpha_2 \pm \sin\alpha_2\cos\alpha_1 = \sin(\alpha_1 \pm \alpha_2)$. Therefore:

$$\Pi_{unit} = \frac{1}{4n}\Big\{\big[(n+1)^2\cos(\alpha_1+\alpha_2) - (n-1)^2\cos(\alpha_1-\alpha_2)\big]\mathbb{1} + i\big[(n+1)^2\sin(\alpha_1+\alpha_2)$$

$$+ (n-1)^2 \sin(\alpha_1-\alpha_2)\big]\sigma_1 - 2(n^2-1)\sin\alpha_1\sin\alpha_2\,\sigma_1 - 2(n^2-1)\sin\alpha_1\cos\alpha_2\,\sigma_2\Big\}$$

For the last layer, $\Pi' = \mathbb{I}_{1\to2}\,\mathbb{P}_1\,\mathbb{I}_{2\to1}$ since there is no air gap to follow the last dielectric layer. Hence, $\Pi' = \mathbb{P}_2^{-1}\,\Pi_{unit}$ and:

$$\Pi_{stack} = \mathbb{P}_2^{-1}\,\Pi^N = \mathbb{P}_2^{-1}(\mathbb{P}_2\,\mathbb{I}_{1\to2}\,\mathbb{P}_1\,\mathbb{I}_{2\to1})^N = \mathbb{P}_2^{-1}(\mathbb{P}_2\,\mathbb{I}_{1\to2}\,\mathbb{P}_1$$

$$\mathbb{I}_{2\to1})^N\,\mathbb{P}_2\,\mathbb{P}_2^{-1} = (\mathbb{P}_2^{-1}\,\mathbb{P}_2\,\mathbb{I}_{1\to2}\,\mathbb{P}_1\,\mathbb{I}_{2\to1}\,\mathbb{P}_2)^N\,\mathbb{P}_2^{-1} \Rightarrow$$

$$\Pi_{stack} = \underbrace{(\mathbb{I}_{1\to2}\,\mathbb{P}_1\,\mathbb{I}_{2\to1}\,\mathbb{P}_2)}_{\Pi \text{ in the book}}{}^N\,\mathbb{P}_2^{-1} = \boxed{\Pi^N(\cos\alpha\,\mathbb{1} - i\sin\alpha_2\,\sigma_3)}$$

(has to do with its choice of unit)

(b) For $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, we have:
($\frac{1}{4}$ wavelength thickness)

$$\Pi = \frac{1}{4n}\big[-(n+1)^2 - (n-1)^2\big]\mathbb{1} + \frac{1}{2n}(n^2-1)\sigma_1 = -\frac{(n^2+1)}{2n}\mathbb{1} - \frac{n^2-1}{2n}\sigma_1$$

Note that:

$$e^{\lambda\sigma_1} = \left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots\right)\mathbb{1} + \left(\lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \dots\right)\sigma_1 = \cosh\lambda\,\mathbb{1} + \sinh\lambda\,\sigma_1$$

For $\lambda = \ln n$, we have:

$$\cosh\lambda = \frac{e^{\lambda}+e^{-\lambda}}{2} = \frac{n^2+1}{2n}, \quad \sinh\lambda = \frac{e^{\lambda}-e^{-\lambda}}{2} = \frac{n^2-1}{2n}$$

Thus:

$$\boxed{\Pi = -e^{-\lambda\sigma_1}}$$

And:

$$\det(\Pi_{stack}) = \det(\Pi^N \mathbb{P}_2^{-1}) = (\det\Pi)^N(\det\mathbb{P}_2)^{-1} = (-1)^{2N} e^{-\ln n \, Tr\sigma_1}$$

Here, we have used $\det(e^A) = e^{Tr A}$. Sinc $Tr\sigma_1 = 0$, we have:

$$\det(\Pi_{stack}) = 1$$

On the other hand:

$$(\Pi_{stack})_{22} = (-1)^N \left(e^{-N\ln n\,\sigma_1}\right)_{22} = (-1)^N \cosh(N\ln n)$$

There fore:

$$\boxed{\left|\frac{E_{trans}}{E_{inc}}\right|^2 = \frac{1}{\cosh^2(N\ln n)} = \frac{1}{(n^N+n^{-N})^2} \xrightarrow[N\to\infty]{} 4n^{-2N} = 4e^{-N\ln n\, 2}}$$